Multiparticle Threshold Amplitudes Exponentiate in Arbitrary Scalar Theories

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Abstract

Threshold amplitudes are considered for n-particle production in arbitrary scalar theory. It is found that, like in ϕ^4, leading-n corrections to the tree level amplitudes, being summed over all loops, exponentiate. This result provides more evidence in favor of the conjecture on the exponential behavior of the multiparticle amplitudes.

1. Considerable interest has been attracted in recent years to the issue of multiparticle production both in perturbative and non-perturbative regimes in weakly coupled scalar theories (for reviews, see [1, 2]). This problem has been initiated by the qualitative observation [3, 4] that in the $\lambda\phi^4$ theory the amplitudes of the processes of creation of a large number of bosons by a few initial particles exhibit factorial dependence on the multiplicity of the final state. The reason is that the number of tree graphs contributing to the amplitude of creation of n particles grows as $n!$. At $n \sim 1/\lambda$ this factor is sufficient to compensate the suppression due to the small coupling constant, and tree level amplitudes become large.

By now, numerous perturbative results in different scalar models have been obtained that confirm the factorial growth of the tree level amplitudes (see, e.g., refs. [5, 6, 7, 8, 9, 10, 11]) and, what is more important, suggest the exponential behavior of multiparticle cross sections [10],

$$\sigma(E,n) \sim \exp\left(\frac{1}{\lambda}F(\lambda n, \epsilon)\right) \tag{1}$$

where $\epsilon = (E - nm)/(nm)$ is the typical kinetic energy of outgoing particles, E is the energy of an initial state, and λ is the typical coupling constant of the theory. Moreover, there exist several results [12] on the first loop corrections at threshold to the tree level amplitudes A_{tree}. It turns out that these corrections are of order $\lambda n^2 A_{\mathrm{tree}}$ and are comparable with the tree level contributions even at $\lambda n < 1$. So, to obtain the correct expressions for the multiparticle amplitudes, one has to take into account all loops. Presently there are two realistic approaches to calculate all leading loop contributions. The first approach has been suggested in models with $O(N)$ symmetry and in models with softly broken $O(N)$ symmetry in the regime $N \to \infty$ [13, 14]. The result coincides with the tree level one with all parameters (masses and coupling constants) replaced by their renormalized values. The second approach has been developed in the context of $\lambda\phi^4$ theory (both with broken and unbroken reflection symmetry) for various initial states [10, 15] with the result that the threshold amplitude is

$$A_{\mathrm{all-loop}}(\epsilon = 0) = A_{\mathrm{tree}} \mathrm{e}^{B\lambda n^2 + O(\lambda n,\, \lambda^2 n^3)} \tag{2}$$

where B is some known numerical constant. It is worth noting that both results support the conjecture of the exponential behavior of the multiparticle cross section (1) with loop effects included.

The purpose of this letter is to demonstrate that the exponential behavior (2) of the threshold amplitudes is inherent in all scalar theories (with constant B being model-dependent) and is not a feature of $\lambda\phi^4$ theory only. We will see, in fact, that the technique developed in refs. [10, 15] can be generalized to any scalar theory in a straightforward way.

2. Let us consider a theory of one scalar field with the action (we set the mass of the scalar boson equal to one)

$$S = \int d^{d+1}x \left(\frac{(\partial_\mu \phi)^2}{2} - \frac{\phi^2}{2} - V(\phi, \{\lambda\}) \right) \tag{3}$$

where $\{\lambda\} = \lambda_3,\ \lambda_4, \ldots$ is the set of coupling constants and

$$V(\phi, \{\lambda\}) = \sum_{k=3} \lambda_k \phi^k$$

In what follows we will assume that the theory can be characterized by a unique weak coupling parameter λ_0, so that the action is proportional to λ_0^{-1} after an appropriate rescaling. This means that

$$\lambda_{k+1} \sim \lambda_k^{\frac{k-1}{k-2}} \sim \lambda_0^{\frac{k-1}{2}} \tag{4}$$

Let us consider the process of creation of n real bosons at threshold (with $(d+1)$-momenta equal to $(1, \mathbf{0})$) by one virtual particle with $(d+1)$-momentum $(n, \mathbf{0})$ in the regime $\{\lambda\} \to 0$, $\lambda_k^{\frac{2}{k-2}} n =$fixed, $\lambda_k^{\frac{2}{k-2}} n \ll 1$. This regime means, in particular, that we will be interested in leading-n behavior in each order of λ_0.

As shown by Brown [7], the tree level amplitude of the process can be obtained by making use of the generating function,

$$A^{\mathrm{tree}}_{1\to n} = \frac{\partial^n \phi_0(z(t))}{\partial z(t)^n}\Big|_{z=0} \tag{5}$$

where ϕ_0 is the classical solution of the spatially homogeneous (due to the threshold kinematics) field equation

$$\partial_t^2 \phi_0 + \phi_0 + V'(\phi_0) = 0 \tag{6}$$

with the boundary condition

$$\phi_0(t \to \infty) = z(t) + \ldots \equiv z_0 \mathrm{e}^{it} + \ldots \tag{7}$$

where dots denote the terms suppressed by λ_0. It is convenient to introduce new euclidean time variable

$$\tau = it + \ln z_0 + C$$

In the generic case the constant C can be chosen in such a way that ϕ_0 is singular at $\tau = 0$,

$$\phi_0(\tau \to 0) = \phi_1(\tau) + \ldots \tag{8}$$

where $\phi_1(\tau)$ is the leading singular term and dots mean less singular ones. One can easily find by scaling that $\phi_1 \sim 1/\sqrt{\lambda_0}$. The leading singularity of ϕ_0 determines the leading-n behavior of the tree level amplitudes. Indeed, making use of the Cauchy theorem one finds

$$A^{\mathrm{tree}}_{1\to n} = \frac{n!}{2\pi i} \oint \frac{d\xi}{\xi^{n+1}} \phi_0(\xi) \simeq n! \int\limits_{C}^{2\pi i + C} d\tau \mathrm{e}^{-n\tau + nC} \phi_1(\tau) \sim n! \lambda_0^{\frac{n-1}{2}} a(n) \tag{9}$$

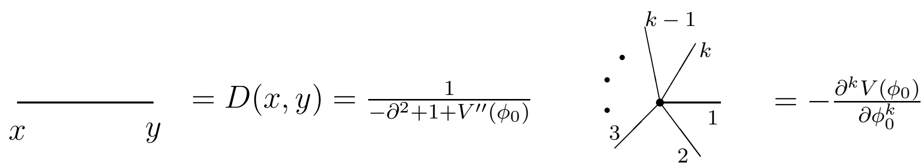


Figure 1: Feynman rules in the classical background ϕ_0.

where $a(n)$ is a function that weakly depends on n. For example, $a \sim n^{m-1}$ for $\phi_1 \sim 1/\tau^m$. It is worth noting that the tree level amplitude grows as $n!$ when $n \to \infty$ in almost any scalar theory (see, however, ref. [16]).

Now let us concentrate on the loop corrections to the tree level amplitude. In complete analogy to the tree level case we will use the generating function formalism. The full amplitude is given by the following formula

$$A_{1\to n}^{\mathrm{all-loop}} = \frac{\partial^n \langle\phi\rangle_{\phi_0}}{\partial z(t)^n}\Big|_{z=0} \tag{10}$$

where the expectation value $\langle\phi\rangle$ is calculated in the classical background ϕ_0. So, extracting the quantum part, $\phi = \phi_0 + \tilde{\phi}$, one can evaluate $\langle\tilde{\phi}\rangle$ by perturbation theory. The corresponding Feynman rules are presented in Fig. 1 and diagram representation of $\langle\tilde{\phi}\rangle$ is shown in Fig. 2.

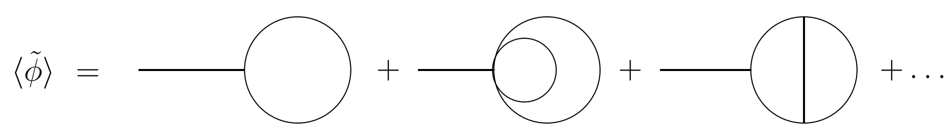


Figure 2: The diagram representation of $\langle\tilde{\phi}\rangle$ in the background field ϕ_0.

Let us now study more closely the properties of the Euclidean propagator $D(x,x')$ in the classical background ϕ_0. The propagator satisfies the following equation ($\partial_0 \equiv \partial_\tau$),

$$(-\partial_\mu^2 + 1 + V''(\phi_0))D(x,x') = \delta^{d+1}(x-x') \tag{11}$$

and decays as $\tau \equiv x_0 \to \pm\infty$. It is convenient to write the propagator in mixed coordinate-momentum representation

$$D(x,x') = \int \frac{d^d p}{(2\pi)^d} \mathrm{e}^{i\mathbf{p}(\mathbf{x}-\mathbf{x}')} D_{\mathbf{p}}(\tau,\tau')$$

One writes

$$D_{\mathbf{p}}(\tau,\tau') = \frac{1}{W_{\mathbf{p}}}(f_1^\omega(\tau)f_2^\omega(\tau')\theta(\tau'-\tau) + f_2^\omega(\tau)f_1^\omega(\tau')\theta(\tau-\tau')) \tag{12}$$

where $f_1^\omega(\tau)$ and $f_2^\omega(\tau)$ are two linearly independent solutions to the homogeneous equation

$$(-\partial_\tau^2 + \omega^2 + V''(\phi_0))f(\tau) = 0 \tag{13}$$

with $\omega = \sqrt{\mathbf{p^2} + 1}$. $f_1^\omega(\tau)$ and $f_2^\omega(\tau)$ tend to zero as $\tau \to -\infty$ and $\tau \to \infty$, respectively. Note, that $\phi_0(\tau \to \pm\infty) \to 0$, $V''(\phi_0(\tau \to \pm\infty)) \to 0$, and, therefore, $f_1^\omega(\tau \to -\infty) \to \mathrm{e}^{\omega\tau}$, $f_2^\omega(\tau \to \infty) \to \mathrm{e}^{-\omega\tau}$. Finally in eq. (12) we introduced the notation

$$W_{\mathbf{p}} = f_1' f_2 - f_2' f_1 \tag{14}$$

$W_{\mathbf{p}}$ is the Wronskian which does not depend on τ.

Let us now concentrate on the behavior of f_1, f_2, and $D_{\mathbf{p}}(\tau,\tau')$ at $\tau \to 0$. One first notes that $V''(\phi_0(\tau \to 0)) = \phi_1'''/\phi_1'$ due to the field equation (6). Generically, both f_1 and f_2 are singular at $\tau = 0$, and one can normalize them in such a way that

$$f_1(\tau) = f_2(\tau) = \phi_1'(\tau) + \ldots \tag{15}$$

in the sense of the leading singularity at $\tau \to 0$. However, equation (15) means that f_1 and f_2 are linearly dependent at $\tau \to 0$. Therefore, it is convenient to introduce the new basis

$$f = \frac{f_1 + f_2}{2}; \quad g = \frac{f_1 - f_2}{2}$$

These functions have different behavior near $\tau = 0$: while f is most singular, $f(\tau \to 0) = \phi_1'(\tau)$, the function g is less singular or regular at this point. To study the behavior of g at $\tau = 0$ one makes use of the independence of $W_{\mathbf{p}}$ on τ. From this condition one obtains

$$g(\tau \to 0) = \frac{W_{\mathbf{p}}}{2}\phi_1(\tau)' \int_0^\tau \frac{d\xi}{(\phi_1(\xi)')^2} \tag{16}$$

In terms of f and g, the propagator can be represented in the following form,

$$D(\tau,\tau';\mathbf{p}) = D_0(\tau,\tau';\mathbf{p}) + D_1(\tau,\tau';\mathbf{p})$$

where

$$D_0(\tau,\tau';\mathbf{p}) = \frac{1}{W_{\mathbf{p}}} f(\tau) f(\tau')$$

$$D_1(\tau,\tau';\mathbf{p}) = \frac{1}{W_{\mathbf{p}}}[\epsilon(\tau-\tau')(f(\tau)g(\tau') - f(\tau')g(\tau)) - g(\tau)g(\tau')]$$

D_0 contains the strongest singularity of D, while D_1 is less singular than D_0. Note that $D_0(\tau,\tau')$ factorizes

$$D_0(\tau,\tau') \simeq \frac{1}{W_{\mathbf{p}}}\phi_1(\tau)'\phi_1(\tau')'$$

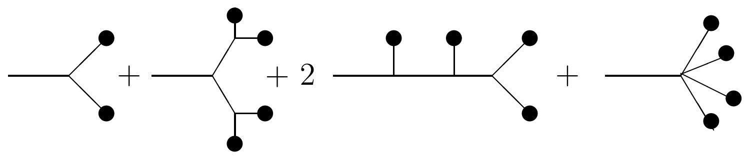
$\tilde{\phi}^{\text{lead}} =$  $+ \ldots$

Figure 3: Graphs contributing to the leading singularity of $\tilde{\phi}$.

while the leading term in D_1 at τ, $\tau' \to 0$ does not depend on $\mathbf{p}$, see eq. (16). These two properties show remarkable similarity to the $\lambda\phi^4$-theory [10] and, as in the $\lambda\phi^4$-theory, will be extensively used in what follows.

Now we are ready to proceed to the analysis of the loop corrections. At first sight, to find the leading singular term in each loop one has to replace all propagators D by D_0. This is, however, not correct. The reason is that $D_0(\tau, \tau')$ is smooth at $\tau = \tau'$, so that the contribution proportional, for instance, to D_0^2 is less singular than the contribution that comes from the product $D_0 D_1$. This is precisely the argument of ref. [10]. It is straightforward to check that the leading singular term in each loop can be obtained by replacement of D by D_0 and D_1 in the following way. First, the obtained graphs should not contain closed loops consisting of D_1 only. Second, these graphs should not factorize, in spite of the factorization of D_0. In particular, the number of D_0's should coincide with the number of loops.

After this procedure, the integration over internal loop momenta becomes trivial because of the fact that the entire dependence on $\mathbf{p}_1, \ldots, \mathbf{p}_k$ (k is the number of loops) is through the product $\prod_{i=1}^{k} W_{\mathbf{p}_i}$. So, after integrating over momenta one finds that the contribution from the k-th loop is proportional to b^k, where

$$b = \frac{1}{2} \int \frac{d^d p}{(2\pi)^d} \frac{1}{W_{\mathbf{p}}} \tag{17}$$

By a simple counting argument one can see that $b \sim \lambda_0$, $b = \lambda_0 B$. Furthermore, it follows from eqs. (13), (14), (15) that the Wronskian grows as $\phi_1(\tau)'' \phi_1(\tau)'|_{\tau=1/\omega}$ when ω tends to infinity. So, at $d \leq 3$ the integral (17) is generically convergent in the ultraviolet, with an exception of the case $d = 3$ and $\lim_{\tau \to 0} \phi_1 \tau^m = 0$ for any $m > 0$.

The procedure described above can be represented graphically. In Fig. 3 several so called bullet [10] graphs are shown. These diagrams correspond to ones presented in Fig. 2. Let us explain some details concerning Fig. 3. First, each line that ends at a bullet corresponds to a factor $(2b)^{1/2}\phi_1'$. Second, each pair of bullets corresponds to an internal line in Fig. 2 which have been replaced by D_0. Because of the factorization of D_0, the internal line can be cut and reduced to two bullets. Third, all bullet diagrams are tree and connected. Therefore the problem of calculation of loop contributions reduces to the summation of certain tree graphs. To perform this calculation, we note that the bullet diagrams have the same form as the graphs in an effective theory where the condensate ϕ_0 is shifted by $(2b)^{1/2}\phi_1'$, with the only

difference in the symmetry factor: each graph with $2k$ external lines ending at bullets must be multiplied by a factor of $(2k)!/(2^k k!)$. So, we proceed follows. We search for a solution to the classical field equation (which is equivalent to the summation of tree graphs) near the singularity,

$$\partial_\tau^2 \phi_{\rm cl} - V'(\phi_{\rm cl}) = 0 \tag{18}$$

(the mass can be neglected) which at small λ_0 has the form

$$\phi_{\rm cl} = \phi_{\rm l} + \sqrt{2b}\phi_{\rm l}' + O(\lambda_0) \tag{19}$$

The solutions of eq. (18) are known,

$$\phi_{\rm cl} = \phi_{\rm l}(\tau + \alpha) \tag{20}$$

again in the sense of the leading singularity. Comparing eqs. (19) and (20) we obtain $\alpha = \sqrt{2b}$ and, therefore,

$$\phi_{\rm cl} = \sum_{i=0}^{\infty} \frac{(2b)^{\frac{i}{2}}}{i!} \frac{\partial^i \phi_{\rm l}(\tau)}{\partial \tau^i} \tag{21}$$

The i-th term of this series corresponds to the sum of tree graphs ending at i bullets. To recover the generating function $\langle\phi\rangle$ one should omit all terms in eq. (21) with odd i and multiply each term with even i, $i = 2k$, by the factor $(2k)!/(2^k k!)$. In this way we obtain

$$\langle\phi\rangle_{\phi_0} = \sum_{k=0}^{\infty} \frac{b^k}{k!} \frac{\partial^{2k} \phi_{\rm l}(\tau)}{\partial \tau^{2k}} \tag{22}$$

Substituting this formula into eq. (10), differentiating by means of the Cauchy theorem, and using eq. (9) we finally get

$$A_{\rm all-loop}(\epsilon = 0) = A_{\rm tree} {\rm e}^{bn^2} \tag{23}$$

Recall that b is of order λ_0.

Therefore, we have shown that the exponentiation of loop corrections is inherent in all scalar theories. The technique can be straightforwardly generalized to the case of the initial state with several particles [15] and to scalar theories with several fields. However, in exceptional models it may happen that the basic assumption that $\phi_0(\tau)$ is singular at $\tau = 0$ (or, in other words, the singularity of ϕ_0 is not at infinity) is not correct. For example, in the theory of ref. [16] with $V(\phi) = (1+\lambda\phi)^2 \ln^2(1+\lambda\phi)^2/(8\lambda^2) - \phi^2/2$ one finds [16]

$$\phi_0(t) = \frac{1}{\lambda}\left({\rm e}^{\lambda z(t)} - 1\right)$$

so that $A^{\rm tree}_{1\to n} = \lambda^{n-1}$ – the tree level amplitudes do not grow factorially! The reason, of course, is that the solution ϕ_0 has a singularity only at $\tau \equiv it + \ln \lambda z_0 \to \infty$. Moreover, more singular terms, for instance ϕ_0^2 or ϕ_0', contribute to the amplitude in the same order as ϕ_0, so the technique described in this letter can not be applied to theories of this type.

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